Solution to "Paper" Homework #11

Section 6.1 - Prob 10

$$4x + y^2 = 12, \quad x = y$$

You can see that it is much easier to express x as a function of y, since, by solving the first equation for x, we have

$$x = -\frac{1}{4}y^2 + 3\,.$$

To find points of intersections, let us solve:

$$y = -\frac{1}{4}y^2 + 3$$

and get $y = -6$ and $y = 2$. Now, compute:

$$A = \int_{-6}^{2} \left(-\frac{1}{4}y^2 + 3 - y\right) dy = \frac{64}{3}\,.$$

Section 6.2 - Problem 8

$$y = \frac{1}{4}x^2, \quad x = 2, \quad y = 0; \quad \text{about the } y\text{-axis}$$

Rotating the region around the y-axis will generate a solid looking like a cylinder partially "eaten up" by a hollowed "solid parabola" (you will call this "paraboloid" in Math 122). For this situation, it is easier to cut the solid in pieces so that the cross sections are perpendicular (making a 90° angle with) to the y-axis. Notice that these cross sections take the form of a washer, so it is obvious that we are using the *washer method*. For this purpose, we need to identify the radii of the inner and outter circles which, of course, can be best expressed as functions of y (think about it). The inner radius corresponds to the parabola curve given as $y = \frac{1}{4}x^2$; but we are interested in this as a function of y, so we rewrite (by solving for x) as $x = 2\sqrt{y} = r(y)$ (calling this function $r(y)$). The outter radius stays constant throughout the calculation, which is $x = 2 = R(y)$ (calling this function $R(y)$). Notice that the region given can be traced completely with y going from 0 to 1. Hence, for each y in $[0, 1]$, the cross section (the washer) has the area $A(y) = \pi[(R(y))^2 - (r(y))^2] = \pi[(2)^2 - (2\sqrt{y})^2] = \pi(4-4y)$. So we can calculate the volume by:

$$V = \int_0^1 A(y)dy = \int_0^1 \pi(4 - 4y)dy = 2\pi\,.$$

Section 6.3 - Prob 6

$$y = 3 + 2x - x^2, \quad x + y = 3$$

To begin, we need to know exactly the region bounded by the given curves by finding points of intersections. The first equation presents y as a fuction of x, so let us rewrite the second equation as a function of x, i.e. $y = -x + 3$. Now, set $3 + 2x - x^2 = -x + 3$ and solve for x, we have $x = 0$ and $x = 3$. So the points of intersections are $(0, 3)$ and $(3, 0)$. For easier reference, call the upper curve $f(x) = 3 + 2x - x^2$ and the lower curve $g(x) = -x + 3$.

Since we are using the *shell method*, we need to identify exactly what these shells are. First, since the rotation is around the y-axis, the cylindrical shells must be symmetric around it as well. Remember, for each cylinder shell, there are two basic quantities: the *height* and the *radius*. Since the solid is generated as x going from 0 to 3 , each cylinder has a radius of x and a height of the distance from the lower curve to the upper curve, that is $f(x) - g(x) = (3 + 2x - x^2) - (-x + 3) = 3x - x^2$. So each shell will have an "infinitesimal volume" of $2\pi \times$ radius $\times$ height $= 2\pi x(3x - x^2)$. Hence we can integrate this from 0 to 3 to get the volume

$$V = \int_0^3 2\pi x(3x - x^2)dx = \frac{27\pi}{2}\,.$$

Section 6.4 - Prob 8

$$y^2 = 4(x+4)^3, \quad 0 \le x \le 2, \quad y > 0$$

Compute for y:

$$y = \pm\sqrt{4(x+4)^3} = \pm 2(x+4)^{3/2}\,.$$

Since the problem only asks for the curve for which $y > 0$, we exclude the negative option, so

$$y = +2(x+4)^{3/2}\,,$$

which is a function of x, hence we can use the formula (2) on page 457 to compute the length:

$$\begin{aligned} L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx &= \int_0^2 \sqrt{1 + \left(3\sqrt{x+4}\right)^2} \\ &= \int_0^2 \sqrt{9x + 37}dx = \frac{110\sqrt{55} - 74\sqrt{37}}{27} \approx 13.5429\,. \end{aligned}$$

Section 6.5 - Prob 4

$$f(\theta) = \sec^2(\theta/2), \quad [0, \pi/2]$$

The average value is calculated by:

$$\frac{1}{b-a}\int_a^b f(\theta)d\theta = \frac{1}{(\pi/2) - 0}\int_0^{\pi/2} \sec^2(\theta/2)d\theta = \frac{4}{\pi}\,.$$