Solution to "Paper" Homework #5

Section 3.4 - Prob 14

$$\begin{aligned} y &= a^3 + \cos^3(x) \\ y' &= (a^3)' + [(\cos(x))^3]' \\ &= 0 + 3 \cdot \cos^2(x)(-\sin(x)) \,. \end{aligned}$$

Section 3.1 - Prob 2

$$\begin{aligned} y &= e^{-2t}\cos(4t) \\ y' &= \left(e^{-2t}\right)' \cos(4t) + e^{-2t}(\cos(4t))' \\ &= e^{-2t}(-2)\cos(4t) + e^{-2t}(-\sin(4t))4 \,. \end{aligned}$$

Section 3.5 - Prob 54

$$x^2 + 4y^2 = 36$$

First we need to find the derivative $\frac{dy}{dx}$; so we have to apply $\frac{d}{dx}$ to both sides of the equations:

$$\begin{aligned} \frac{d}{dx}(x^2 + 4y^2) &= \frac{d}{dx}(36) \\ \frac{d}{dx}(x^2) + 4\frac{d}{dx}(y^2) &= 0 \\ 2x + 4 \cdot 2 \cdot y\frac{dy}{dx} &= 0 \\ 8 \cdot y\frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= \frac{-2x}{8y} \\ \frac{dy}{dx} &= \frac{-x}{4y} \,. \end{aligned}$$

So, if (x_o, y_o) is a point on the graph (in this case the ellipse), then the slope m of the tangent line to the graph at this point is given by

$$m = \left.\frac{dy}{dx}\right|_{(x_o, y_o)} = \frac{-x_o}{4y_o} \,.$$

Hence, the equation of the tangent line is

$$\begin{aligned} y - y_o &= m(x - x_o) \\ y - y_o &= \frac{-x_o}{4y_o}(x - x_o) \,. \qquad (\dagger) \end{aligned}$$

But since the tangent line passes through the point $(12, 3)$ we have:

$$\begin{aligned}
3 - y_o &= \frac{-x_o}{4y_o}(12 - x_o) \\
12y_o - 4y_o^2 &= -12x_o + x_o^2 \\
12(x_o + y_o) &= x_o^2 + 4y_o^2 \\
12(x_o + y_o) &= 36 \\
x_o + y_o &= 3 \\
\sqrt{36 - 4y_o^2} + y_o &= 3 \\
\sqrt{36 - 4y_o^2} &= 3 - y_o \\
36 - 4y_o^2 &= 9 - 6y_o + y_o^2 \\
5y_o^2 - 6y_o - 27 &= 0\,.
\end{aligned}$$

Solve the equation for y_o, we then have:

$$y_o = -\frac{9}{5} \qquad \Longrightarrow \qquad x_o = 3 - y_o = \frac{24}{5}$$

and

$$y_o = 3 \qquad \Longrightarrow \qquad x_o = 3 - y_o = 0\,.$$

By substituting these pairs of (x_o, y_o) into the equation marked with a $(\dagger)$ above, we have the results (after simplifications):

$$y = 3 \qquad \text{and} \qquad y = \frac{2}{3}x - 5\,.$$

Section 3.6 - Prob 34 Let

$$f(x) = y = 3\arccos(\frac{x}{2})\,.$$

Compute the derivative:

$$f'(x) = 3\frac{-1}{\sqrt{1 - (\frac{x}{2})^2}} \cdot \frac{1}{2}\,.$$

Compute the slope of the tangent line

$$m = f'(1) = -\sqrt{3}\,.$$

Now use the point-slope form:

$$\begin{aligned}
y - y_o &= m(x - x_o) \\
y - \pi &= -\sqrt{3}(x - 1)\,.
\end{aligned}$$

Section 3.7 - Prob 4

$$f(x) = \ln(\sin^2 x)$$

$$
\begin{aligned}
f'(x) &= \left[\ln(\sin^2 x)\right]' \\
&= \frac{1}{\sin^2 x}\left(\sin(x)^2\right)' \\
&= \frac{1}{\sin^2 x}\cdot 2\sin(x)\cos(x) \\
&= 2\cot(x)\,. \qquad \text{(optional)}
\end{aligned}
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