Section 5.5

$$\frac{d}{dx}\ln|x| = \frac{1}{x} \qquad (\text{provided } x \neq 0)$$

proof | suppose $f(x) = \ln(x)$

$x = e^{f(x)}$

derivative of both sides

$$\frac{d}{dx}[x] = \frac{d}{dx}\left[e^{f(x)}\right]$$

$$1 = e^{f(x)} f'(x)$$

move $e^{f(x)}$ over

$$\frac{1}{e^{f(x)}} = f'(x)$$

but $e^{f(x)} = x$

$$\frac{1}{x} = f'(x)$$

EX $f(x) = x\ln x$ $\qquad g(x) = \dfrac{\ln x}{x}$

$$f'(x) = \frac{d}{dx}[x\ln x]$$

product rule

$$= \frac{d}{dx}[x](\ln x) + \frac{d}{dx}[\ln x](x)$$

$$= (1)(\ln x) + \left(\frac{1}{x}\right)(x)$$

$$= (\ln x) + 1$$

$$g'(x) = \frac{d}{dx}\left[\frac{\ln x}{x}\right]$$ quotient rule

$$= \frac{\frac{d}{dx}[\ln x](x) - \frac{d}{dx}[x](\ln x)}{x^2}$$

$$= \frac{(1/x)(x) - (1)(\ln x)}{x^2}$$

$$= \frac{1-\ln x}{x^2}$$

$$\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)} \qquad \text{if } f(x) > 0$$

EX $f(x) = \ln(x^2+1)$

$$f'(x) = \cancel{x^2+1} \qquad \frac{f'(x)}{f(x)}$$

$$= \frac{2x}{x^2+1}$$

EX $f(x) = \ln\left[(x^2+1)(x^3+2)^6\right] = \ln(x^2+1) + \ln(x^3+2)^6$

$$= \ln(x^2+1) + 6\ln(x^3+3)$$

$$f'(x) = \frac{d}{dx}\left[\ln(x^2+1)\right] + 6\frac{d}{dx}\left[x^3+3\right]$$

$$\cancel{f'(x) = \frac{\frac{d}{dx}\left[(x^2+1)(x^3+2)^6\right]}{(x^2+1)(x^3+2)^6}}$$

$$= \cancel{\frac{\frac{d}{dx}\left[x^2+1\right](x^3+2)^6 + \frac{d}{dx}\left[(x^3+2)^6\right](x^2+1)}{(x^2+1)(x^3+2)^6}} = \frac{2x}{x^2+1} + 6\frac{3x^2}{x^3+3}$$

Top $\cancel{(2x)(x^3+2)^6 + 6(x^3+2)^5(3x^2)(x^2+1)}$ $= \dfrac{2x}{x^2+1} + \dfrac{18x^2}{x^3+3}$

Top $\cancel{(x^3+2)^5\left[2x(x^3+2) + \underbrace{6(3x^2)}_{18}(x^2+1)\right]}$

$\cancel{(x^3+2)^5(2x^4+2x+18x^4+18x^2)}$

EX $g(t) = \ln\left(t^2 e^{-t^2}\right)$

$$g(t) = \ln(t^2) + \ln\left(e^{-t^2}\right)$$
$$= 2\ln(t) + -t^2$$
$$= 2\ln(t) - t^2$$
$$g'(t) = 2\frac{1}{t} - 2t \qquad \left(= \frac{2(1-t^2)}{t}\right.$$

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logarithmic diferentiation

EX $Y = X(X+1)(X^2+1)$

$$\ln Y = \ln\left(X(X+1)(X^2+1)\right)$$
$$= \ln(X) + \ln(X+1) + \ln(X^2+1)$$
$$\frac{d}{dX}\ln Y = \frac{d}{dX}\left[\ln(X) + \ln(X+1) + \ln(X^2+1)\right]$$
$$\frac{Y'}{Y} = \frac{1}{X} + \frac{1}{X+1} + \frac{2X}{X^2+1}$$
$$Y' = Y\left(\frac{1}{X} + \frac{1}{X+1} + \frac{2X}{X^2+1}\right)$$
$$Y' = (X)(X+1)(X^2+1)\left(\frac{1}{X} + \frac{1}{X+1} + \frac{2X}{X^2+1}\right)$$

$$\frac{d}{dX}[\ln Y] = \frac{d}{dX}[\ln f(X)] = \frac{f'(X)}{f(X)} = \frac{Y'}{Y}$$

Steps

(1) take ln of both sides
(2) use property of logs to rewrite complicated stuff
(3) take derivative of both sides
(4) solve for Y'

EX

$$Y = X^2 (X-1)(X^2+4)^3$$

$$\ln Y = \ln\left(X^2 (X-1)(X^2+4)^3\right)$$

$$= \ln(X^2) + \ln(X-1) + \ln(X^2+4)^3$$

$$= 2\ln(X) + \ln(X-1) + 3\ln(X^2+4)$$

$$\frac{d}{dX} \ln Y = 2\frac{1}{X} + \frac{1}{X-1} + 3\frac{2X}{X^2+4}$$

$$\frac{Y'}{Y} = \frac{2}{X} + \frac{1}{X-1} + \frac{6X}{X^2+4}$$

$$Y' = Y\left(\frac{2}{X} + \frac{1}{X-1} + \frac{6X}{X^2+4}\right)$$

$$= X^2(X-1)(X^2+4)^3\left(\frac{2}{X} + \frac{1}{X-1} + \frac{6X}{X^2+4}\right)$$