Solution to "Paper" Homework #10

Section 5.6 - Prob 14

$$\int e^{-\theta}\cos(2\theta)d\theta\,.$$

Let:

$$\begin{aligned} u &= \cos(2\theta) & dv &= e^{-\theta}d\theta \\ du &= -2\sin(2\theta)d\theta & v &= \int e^{-\theta}d\theta = -e^{-\theta}\,. \end{aligned}$$

So:

$$\begin{aligned} \int e^{-\theta}\cos(2\theta)d\theta &= uv - \int vdu \\ &= \cos(2\theta)(-e^{-\theta}) - \int(-e^{-\theta})(-2\sin(2\theta))d\theta \\ &= -\cos(2\theta)e^{-\theta} - 2\underbrace{\int e^{-\theta}\sin(2\theta)d\theta}_{I} \end{aligned}$$

Now compute:

$$I = \int e^{-\theta}\sin(2\theta)d\theta\,.$$

Do another integration by part process:

$$\begin{aligned} u &= \sin(2\theta) & dv &= e^{-\theta}d\theta \\ du &= 2\cos(2\theta)d\theta & v &= \int e^{-\theta}d\theta = -e^{-\theta}\,. \end{aligned}$$

We have:

$$I = \sin(2\theta)(-e^{-\theta}) - \int(-e^{-\theta})2\cos(2\theta)d\theta$$

$$\begin{aligned} I &= \sin(2\theta)(-e^{-\theta}) - \int(-e^{-\theta})2\cos(2\theta)d\theta \\ &= -\sin(2\theta)e^{-\theta} + 2\int e^{-\theta}\cos(2\theta)d\theta \end{aligned}$$

Now substitute this result back to the equation above, we have:

$$\begin{aligned} \int e^{-\theta}\cos(2\theta)d\theta &= -\cos(2\theta)e^{-\theta} - 2\left(-\sin(2\theta)e^{-\theta} + 2\int e^{-\theta}\cos(2\theta)\right) \\ \Rightarrow \quad \int e^{-\theta}\cos(2\theta)d\theta &= -\cos(2\theta)e^{-\theta} + 2\sin(2\theta)e^{-\theta} - 4\int e^{-\theta}\cos(2\theta) \\ \Rightarrow \quad 5\int e^{-\theta}\cos(2\theta)d\theta &= -\cos(2\theta)e^{-\theta} + 2\sin(2\theta)e^{-\theta} \\ \Rightarrow \quad \int e^{-\theta}\cos(2\theta)d\theta &= \frac{1}{5}[-\cos(2\theta)e^{-\theta} + 2\sin(2\theta)e^{-\theta}] + C\,. \end{aligned}$$

Section 5.7 - Prob 24

$$\int \frac{x^2+2x-1}{x^3-x}dx\,.$$

Now, let us work on the integrand (" the function of which we want to find the antiderivative"):

$$\frac{x^2+2x-1}{x^3-x} = \frac{x^2+2x-1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

Consider:

$$\frac{x^2+2x-1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}\,.$$

Now, multiplying both the left and right of the equation by the common denominator $x(x-1)(x+1)$, we have:

$$\begin{aligned} x^2+2x-1 &= A(x-1)(x+1) + Bx(x+1) + Cx(x-1) \quad (\dagger) \\ x^2+2x-1 &= (A+B+C)x^2 + (B-C)x - A \end{aligned}$$

By comparing the coefficients on both sides of the equation, we have:

$$\left\{ \begin{array}{rrrcr} A & +B & +C & = & 1 \\ & +B & -C & = & 1 \\ -A & & & = & 1 \end{array} \right. \Rightarrow \left\{ \begin{array}{rcr} A & = & 1 \\ B & = & 1 \\ C & = & -1 \end{array} \right.$$

(Note: a faster way to do this is to subsequently substitute $x = 0, 1, -1$ into the equation marked with a ($\dagger$) above.)
So, we have:

$$\frac{x^2+2x-1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}\,.$$

$$\begin{aligned} \int \frac{x^2+2x-1}{x^3-x}dx &= \int \left(\frac{1}{x} + \frac{1}{x-1} + \frac{-1}{x+1} \right) dx \\ &= \ln|x| + \ln|x-1| - \ln|x+1| + C \end{aligned}$$

Section 5.8 - Prob 4

$$\int_2^3 \frac{1}{x^2\sqrt{4x^2-7}}dx$$

In the reference page 6, the formula closest to this integral is (# 28)

$$\int \frac{du}{u^2\sqrt{a^2+u^2}} = -\frac{\sqrt{a^2+u^2}}{a^2u} + C$$

Notice that the formula still make sense if we replace a^2 by a more general number, let's say, b:

$$\int \frac{du}{u^2\sqrt{b+u^2}} = -\frac{\sqrt{b+u^2}}{bu} + C\,.$$

Now, consider:

$$I = \int \frac{1}{x^2\sqrt{4x^2-7}} = \int \frac{1}{2x^2\sqrt{x^2-\frac{7}{4}}}dx = -\frac{1}{2}\frac{\sqrt{-\frac{7}{4}+x^2}}{-\frac{7}{4}x} + C = \frac{\sqrt{4x^2-7}}{7x} + C\,.$$

(You can see that b in this case is $-\frac{7}{4}$.) So

$$\int_2^3 \frac{1}{x^2\sqrt{4x^2-7}}dx = \left.\frac{\sqrt{4x^2-7}}{7x}\right|_2^3 = \frac{\sqrt{29}}{21} - \frac{3}{14}$$

Section 5.9 - Prob 8

$$\int_0^{1/2} \sin(x^2)dx\,, \quad n = 4$$

$$\Delta x = \frac{1/2-0}{4} = \frac{1}{8}\,.$$

The interval $[0, 1/2]$ is divided into four equal subintervals: $[0, 1/8]$; $[1/8, 1/4]$; $[1/4, 3/8]$; and $[3/8, 1/2]$.

(a) Using the Trapezoidal Rule:

$$\int_0^{1/2} \sin(x^2)dx \approx \frac{\Delta x}{2}[f(0) + 2f(1/8) + 2f(1/4) + 2f(3/8) + f(1/2)] = 0.042743\,.$$

(b) Using the Midpoint Rule:
The midpoints of the subintervals listed above are $1/16, 3/16, 5/16$, and $7/16$ respectively. Hence, by the Mid-point rule:

$$\int_0^{1/2} \sin(x^2)dx \approx \Delta x[f(1/16) + f(3/16) + f(5/16) + f(7/16)] = 0.040850\,.$$

(c) Using the Simpson's Rule:

$$\int_0^{1/2} \sin(x^2)dx \approx \frac{\Delta x}{3}[f(0) + 4f(1/8) + 2f(1/4) + 4f(3/8) + f(1/2)] = 0.041478\,.$$

Section 5.10 - Prob 22

$$\int_1^{\infty} \frac{\ln x}{x^3}dx$$

By performing integration-by-part technique (with $u = \ln(x)$ and $dv = x^{-3}dx$) we have:

$$\int \frac{\ln x}{x^3} = -\frac{1}{2}\frac{\ln(x)}{x^2} - \frac{1}{4}\frac{1}{x^2}$$

Consider:

$$\begin{aligned}
\int_1^{\infty} \frac{\ln x}{x^3} &= \lim_{t\to\infty} \int_1^{t} \frac{\ln x}{x^3} dx = \lim_{t\to\infty} \left[-\frac{1}{2}\frac{\ln(x)}{x^2} - \frac{1}{4}\frac{1}{x^2} \right] \Bigg|_1^t \\
&= \lim_{t\to\infty} \left[-\frac{\ln(t)}{2t^2} - \frac{1}{4t^2} \right] - \left[-\frac{\ln(1)}{2\cdot 1^2} - \frac{1}{4\cdot 1^2} \right] \\
&= \lim_{t\to\infty} \left[-\frac{\ln(t)}{2t^2} - \frac{1}{4t^2} + \frac{1}{4} \right] \\
&= \underbrace{-\lim_{t\to\infty} \frac{\ln(t)}{2t^2}}_{L'Hopital} - \lim_{t\to\infty} \frac{1}{4t^2} + \frac{1}{4} = \frac{1}{4}\,.
\end{aligned}$$