Solution to "Paper" Homework #12

Section 6.6 - Prob 12 After raising one end of the chain to the height of 6 m, we observe that the chain has two parts: the "hanging" part which is 6-meter long, and the "lying" part which lies on the ground. Notice that we have applied work to the "hanging" part, but not to the part on the ground. So, in this calculation, we only consider the former. Now, for this particular part, imagine that we can divide it in many (let say, multi-million) equal pieces; each piece has a certain length Δx (which determines the mass $m_i = \rho \cdot \Delta x$—where ρ is the mass density— and consequently the weight $P_i = m_i \cdot g$) and is at a certain height x_i. Remember from basic Physics theory, in order to lift this piece from the ground to this height, we must apply a work $W_i = P_i \cdot x_i = m_i \cdot g \cdot x_i = \rho \cdot \Delta x \cdot g \cdot x_i$. Hence, if we have divided this part into n pieces, the approximate work done can be expressed by:

$$W \approx \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} \rho \cdot \Delta x \cdot g \cdot x_i = \sum_{i=1}^{n} \rho \cdot g \cdot x_i \cdot \Delta x$$

However, this is just an approximation of the work done. To make it exact, we divide the "hanging" part into infinitely many pieces, i.e. taking $n \to \infty$. So the summation above becomes this integral:

$$W = \int_0^6 \rho \cdot g \cdot x \, dx$$

The problem states that the chain is 10 meter-long, with a mass of 80 kg. So the mass density is $\rho = 80/10 = 8$ kg/m. And with $g = 9.8$ m/s^2, we have

$$W = \int_0^6 8 \cdot 9.8 \cdot x \, dx = 1411.2 \, .$$

Section 6.6 - Prob 48

$$y = f(x) = 1/x, \quad y = 0, \quad x = 1, \quad x = 2 \, .$$

Carefully sketch the graph. We can quickly calculate the mass of the region:

$$M = \int_1^2 \rho f(x) dx = \int_1^2 \rho \frac{1}{x} dx = \rho \ln(2) \, .$$

First, let us find M_y, the moment with respect to the y-axis. Recall that if we have n objects, each with a mass m_i and a distance of x_i from the centroid to the x-axis, then the moment with respect to the same axis is given by:

$$\sum_{i=1}^{n} m_i \cdot x_i = m_1 x_1 + m_2 x_2 + \ldots + m_n x_n \, .$$

For this problem, imagine that we divide the region given into n rectangular strips (parallel to the y-axis). Each rectangle, let say, had a distance of x_i away from the y-axis (the same is the distance between the centroid of this rectangle and the y-axis), with a width of Δx, and a height of $f(x_i)$ (hence, the mass is $m_i = \rho \cdot f(x_i) \cdot \Delta x$). So in this setting, the moment can be approximated by:

$$M_y \approx \sum_{i=1}^{n} m_i x_i = \sum_{i=1}^{n} \rho \cdot f(x_i) \cdot \Delta x \cdot x_i = \sum_{i=1}^{n} \rho \cdot x_i \cdot f(x_i) \cdot \Delta x\,.$$

The exact moment is given when the number of rectangles becomes infinity, i.e. when $n \to \infty$, then the summation above becomes the integral

$$M_y = \int_1^2 \rho \cdot x \cdot f(x) dx = \int_1^2 \rho \cdot x \cdot \frac{1}{x} dx = \int_1^2 \rho dx = \rho\,.$$

Now, we compute M_x, the moment with respect to the x-axis. Repeat the same process above, with the same setting of rectangle strips. But this time, for each piece, the distance between the centroid and the x-axis is $y_i = f(x_i)/2$. Hence, an approximation of M_x is given by

$$M_x \approx \sum_{i=1}^{n} m_i y_i = \sum_{i=1}^{n} \rho \cdot f(x_i) \cdot \Delta x \cdot \frac{f(x_i)}{2} = \frac{1}{2} \sum_{i=1}^{n} \rho \cdot [f(x_i)]^2 \cdot \Delta x\,,$$

and when $n \to \infty$, M_x becomes exact, given by

$$M_x = \frac{1}{2} \int_1^2 \rho [f(x)]^2 dx = \frac{1}{2} \int_1^2 \rho \frac{1}{x^2} dx = \frac{1}{4} \rho\,.$$

The coordinate $(\bar{x}, \bar{y})$ of the centroid of the region is given by:

$$\bar{x} = \frac{M_y}{M} = \frac{\rho}{\rho \ln(2)} = \frac{1}{\ln(2)} \approx 1.442695\,,$$

$$\bar{y} = \frac{M_x}{M} = \frac{(1/4)\rho}{\rho \ln(2)} = \frac{1}{4\ln(2)} \approx 0.360674\,.$$

Section 6.7 - Prob 4

$$p(x) = 20 - 0.05x\,.$$

When the sales level is 300, the unit price becomes $p(300) = 5$. The consumer surplus is the "area" between the curve $p(x)$, the line $p = 5$ from $x = 0$ to $x = 300$, given by:

$$\int_0^{300} [p(x) - 5] dx = \int_0^{300} [15 - 0.05x] dx = 2250\,.$$

Section 6.8 - Prob 6

$$f(x) = \begin{cases} kx^2(1-x) & \text{if} \quad 0 \le x \le 1\,, \\ 0 & \text{if} \quad x < 0 \text{ or } x > 1\,. \end{cases}$$

(a) By definition, $f(x)$ is a probability density function if these two conditions are satisfied:

1. $f(x) \ge 0$. In this problem, this is true whenever $k \ge 0$.

2. $\int_{-\infty}^{+\infty} f(x)dx = 1$. We have

$$1 = \int_{-\infty}^{+\infty} f(x)dx = \int_0^1 kx^2(1-x)dx = \frac{k}{12}\,,$$

which implies that $k = 12$.

(b)

$$P(X \geq \frac{1}{2}) = \int_{\frac{1}{2}}^{+\infty} f(x)dx = \int_{\frac{1}{2}}^{1} kx^2(1-x)dx = \int_{\frac{1}{2}}^{1} 12x^2(1-x)dx = \frac{11}{16}\,.$$

(c) The mean is given by

$$\mu = \int_{-\infty}^{+\infty} xf(x)dx = \int_0^1 x \cdot (kx^2(1-x))dx = \int_0^1 x \cdot (12x^2(1-x))dx = \frac{3}{5}\,.$$

Section 6.8 - Prob 7

(a)

$$f(x) = \begin{cases} 0.1 & \text{if} \quad 0 \leq x \leq 10\,, \\ 0 & \text{if} \quad x < 0 \text{ or } x > 10\,. \end{cases}$$

Since

$$\int_{-\infty}^{+\infty} f(x)dx = \int_0^{10} 0.1dx = 1\,,$$

and since $f(x) \geq 0$ for all x, by definition, $f(x)$ is a probability density function.

(b) The mean is given by

$$\mu = \int_{-\infty}^{+\infty} xf(x)dx = \int_0^{10} x \cdot 0.1\, dx = 5\,.$$