Solution to "Paper" Homework #6

Section 3.8 - Prob 17 The fuction that measure the mass of the part of the metal rod x m from the left end point is given by $f(x) = 3x^2$, whose derivative is $f'(x) = 6x$.

(a) $f'(1) = 6$.

(b) $f'(2) = 12$.

(c) $f'(3) = 18$.

The density is highest at the right end, and lowest at the left end.

Section 3.9 - Prob 27

(a) The volume of the cube is given by $V(x) = x^3$. If x increases/decreases *from* x_o (which is the starting value, in this particular case 30 cm) with an amount Δx (in this case $\Delta x = 0.1$ cm), the change in V is approximated by:

$$\Delta V \approx V'(x_o)\Delta x = 3x_o^2\Delta x = 3 \cdot (30)^2 \cdot 0.1 = 270,$$

which, of course, is the maximum error. The relative error is given by

$$\frac{\Delta V}{V} \approx \frac{270}{30^3} = 0.01\,,$$

and the percentage error is 1%

(b) The surface area is given by $A(x) = 6x^2$. The same concepts and procedures apply as above. So we have:

$$\Delta A \approx A'(x_o)\Delta x = 12x_o\Delta x = 12 \cdot 30 \cdot 0.1 = 36\,,$$

which is the maximum error. The relative error is

$$\frac{\Delta A}{A} \approx \frac{36}{6 \cdot 30^2} = 0.006\bar{6}\,,$$

and the percentage error is $0.6\bar{6}$.

Section 4.1 - Prob 3 The area of a square is given by

$$A = x^2$$

and we are interested in the rate of change of A with respect to t (time), i.e. we want to find $\frac{dA}{dt}$. Apply $\frac{d}{dt}$ to both sides of the equation above, we have

$$\begin{aligned}\frac{d}{dt}A &= \frac{d}{dt}x^2\\ \frac{dA}{dt} &= 2x \cdot \frac{dx}{dt}\,.\end{aligned}$$

When the area is $A = x^2 = 16$, $x = 4$. This, together with the clue that $\dfrac{dx}{dt} = +6$ help us find:

$$\frac{dA}{dt} = 2 \cdot 4 \cdot 6 = 48\,.$$

Section 4.2 - Prob 51

$$f(x) = \ln(x^2 + x + 1) \qquad \text{on } [-1, 1]\,.$$

Since this is a closed interval, we can use the techniques outlined in section 4.2. First, we compute the derivative:

$$f'(x) = \frac{1}{x^2 + x + 1} \cdot (2x + 1)\,.$$

$f'(x) = 0$ when $x = -\frac{1}{2}$. Now, compute values of $f(x)$ (*not*) $f'(x)$) at this critical value and the endpoints:

- $f(-\frac{1}{2}) = \ln(\frac{3}{4})\,.$
- $f(-1) = 0\,.$
- $f(1) = \ln(3)\,.$

So at $x = -\frac{1}{2}$, the $f(x)$ has an absolute minimum of $f(-\frac{1}{2}) = \ln(\frac{3}{4})$; and at $x = 1$, $f(x)$ has an absolute maximum of $f(1) = \ln(3)$.

Section 4.2 - Prob 53

$$f(t) = 2\cos(t) + \sin(2t) \qquad \text{on } [0, \pi/2]\,.$$

Compute the derivative $f'(t) = -2\sin(t) + \cos(2t) \cdot 2$.

$$\begin{aligned} f'(t) &= 0 \\ -2\sin(t) + 2\cos(2t) &= 0 \\ -\sin(t) + \cos(2t) &= 0 \\ -\sin(t) + 1 - 2\sin^2(t) &= 0 \\ -2\sin^2(t) - \sin(t) + 1 &= 0 \end{aligned}$$

Solve the quadratic equation for $\sin(t)$, we have

$$\sin(t) = -1 \qquad \text{or} \qquad \sin(t) = \frac{1}{2}\,.$$

Since $\sin(t) \geq 0$ on $[0, \pi/2]$, we choose only choose $\sin(t) = 1/2$, which implies $t = \pi/6$; this is our critical value. Now, evaluate the original function $f(t)$ at this critical value and at the end points, we have:

- $f(0) = 2\,.$
- $f(\pi/6) = \frac{3\sqrt{3}}{2} \approx 2.598\,.$
- $f(\pi/2) = 0\,.$

We conclude that the function (on the specified interval) has an absolute minimum of 0 at $t = \pi/2$, and an absolute maximum of $\frac{3\sqrt{3}}{2}$ at $t = \pi/6$.