**Solution to "Paper" Homework #7**

**Section 4.3 - Prob 16**

$$f(x) = \sqrt{x}e^{-x}\,.$$

Note: the domain of $f(x)$ is $(0, +\infty)$.

(a) Since whether the function increases or decreases depends entirely on the sign of its derivative, so we compute:

$$f'(x) = e^{-x}\left(\frac{1}{2\sqrt{x}} - \sqrt{x}\right) = e^{-x}\,\frac{1-2x}{2\sqrt{x}}\,.$$

Notice that the result has been simplified into a fraction form since we are interested in finding the critical values (those that make $f'(x) = 0$ or $DNE$). Now, find the critical values:

- $f'(x) = 0$ when $x = 1/2$.
- $f'(x)$ is undefined (or does not exist) at $x = 0$.

So we have two critical values: $x = 0$ and $x = 1/2$. Now, perform the sign test on $f'(x)$. Notice that since $e^{-x} > 0$ and $2\sqrt{x} > 0$ on the domain of $f(x)$, the sign of $f'(x)$ only depends on the sign of $1 - 2x$.

| $x$ | $(0, 1/2)$ | $(1/2, +\infty)$ |
|---|---|---|
| sign of $f'(x)$ | $+$ | $-$ |
| Conclusion on $f(x)$ | $\nearrow$ | $\searrow$ |

So the function $\nearrow$ on $(0, 1/2)$; and $\searrow$ on $(1/2, +\infty)$.

(b) From the sign test table, we can see clearly that since $1/2$ is in the domain, the function has a local maximum of $f(1/2) = \sqrt{1/2}\,e^{-1/2}$ at $x = 1/2$.

(c) Whether the function concaves upward or downward depends on the sign of its second derivative. So we have (after some long simplification):

$$f''(x) = e^{-x}\,\frac{4x^2 - 4x - 1}{4x^{3/2}}$$

Now find the values of $x$ that make $f''(x)$ either zero or undefined:

- $f''(x) = 0$ when $4x^2 - 4x - 1 = 0$ which give two roots: $x_1 = \frac{-\sqrt{2}+1}{2}$ (disregarded since it is out of the domain) and $x_2 = \frac{\sqrt{2}+1}{2}$
- $f''(x) = DNE$ when $x = 0$.

Now do the sign test for $f''(x)$ as we did above:

| $x$ | $(0, \frac{\sqrt{2}+1}{2})$ | $(\frac{\sqrt{2}+1}{2}, +\infty)$ |
|---|---|---|
| sign of $f''(x)$ | $-$ | $+$ |
| Conclusion on $f(x)$ | $\bigcap$ | $\bigcup$ |

So the $f$ concaves downwards on $(0, \frac{\sqrt{2}+1}{2})$ and upwards on $(\frac{\sqrt{2}+1}{2}, +\infty)$. And since $\frac{\sqrt{2}+1}{2}$ is in the domain the point $\left(\frac{\sqrt{2}+1}{2}, f\left(\frac{\sqrt{2}+1}{2}\right)\right)$ is the inflection point.

**Section 4.3 - Prob 18**

$$f(x) = \frac{x}{x^2+4}$$

Domain: $D = (-\infty, \infty)$

(a) First Derivative Test:

$$f'(x) = \frac{-x^2+4}{(x^2+4)^2}$$

Now, find the critical values. Since $f'(x)$ is defined everywhere, we only need to find those that make it zero: $f'(x) = 0$ when $x = -2$ or $x = +2$. Perform the sign test:

| $x$ | $(-\infty, -2)$ | $(-2, 2)$ | $(2, +\infty)$ |
|---|---|---|---|
| sign of $f'(x)$ | $-$ | $+$ | $-$ |
| Conclusion on $f(x)$ | $\searrow$ | $\nearrow$ | $\searrow$ |

Since both $-2$ and $2$ are in the domain, the $f(x)$ has a local min. of $f(-2) = -1/4$ at $x = -2$, and a local max. of $f(2) = 1/4$ at $x = 2$.

(b) Repeat the first steps above to find the critical values of $f'(x)$. Now, compute the second derivative:

$$f''(x) = \frac{2x(x^2-12)}{(x^2+4)^3}$$

Evaluate:

- $f''(-2) = 1/16 > 0$ (local min.).
- $f''(2) = -1/16 < 0$ (local max.).

Make the same conclusions as above.

**Section 4.3 - Prob 58**

$$f(x) = e^{-x^2/(2\sigma^2)}$$

(a) As $x \to \pm\infty$, $-x^2/(2\sigma^2) \to -\infty$, so $f(x) \to 0$.

$$f'(x) = \frac{-x}{\sigma^2} e^{-x^2/(2\sigma^2)} .$$

Do a sign test to conclude that $f(x)$ increases on $(-\infty, 0)$ and decreases on $(0, +\infty)$, thus have a local maximum of $f(0) = 1$ at $x = 0$, which is also the absolute maximum in this case. Take the second derivative:

$$f''(x) = \left(\frac{x^2}{\sigma^4} - \frac{1}{\sigma^2}\right) e^{-x^2/(2\sigma^2)}$$

You can conclude, after doing a sign test, that $(-\sigma, e^{-1/2})$ and $(\sigma, e^{-1/2})$ are inflection points.

(b) If we decrease $\sigma$, the curve will be more like of a bell shape (the "width" of the bell gets narrower and narrower) and vice versa.

**Section 4.4 - Prob 8**

$$f(x) = \frac{e^x}{x^2 - 9}$$

1. Domain: $D = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

2. x and y-intercepts:

   - $f(0) = -1/9$.
   - $f(x) = 0$ has no solution, so there is no x-intercept.

3. Symmetry: Since $f(-x) \neq f(x)$ or $-f(x)$, there is no symmetry, neither with respect to the y-axis nor the origin.

4. Asymtotes:

   - $\lim_{x\to -3^-} f(x) = +\infty$.
   - $\lim_{x\to -3^+} f(x) = -\infty$.
   - $\lim_{x\to 3^-} f(x) = -\infty$.
   - $\lim_{x\to 3^+} f(x) = +\infty$.
   - $\displaystyle\lim_{x\to+\infty} \frac{e^x}{x^2-9} =^H \lim_{x\to+\infty} \frac{e^x}{2x} =^H \lim_{x\to+\infty} \frac{e^x}{2} = +\infty$.
   - $\displaystyle\lim_{x\to-\infty} \frac{e^x}{x^2-9} = \lim_{x\to-\infty} \underbrace{e^x}_{\to 0} \cdot \underbrace{\frac{1}{x^2-9}}_{\to 0} = 0$.

5. Increasing/Decreasing: Compute the derivative:

$$f'(x) = \frac{e^x \cdot (x^2 - 2x - 9)}{(x^2-9)^2}.$$

   Perform a sign test (notice that the sign of $f'(x)$ depends entirely on the sign of $x^2 - 2x - 9$, a quadratic function.

| $x$ | $(-\infty, -3)$ | $(-3, 1-\sqrt{10})$ | $(1-\sqrt{10}, 3)$ | $(3, 1+\sqrt{10})$ | $(1+\sqrt{10}, +\infty)$ |
|---|---|---|---|---|---|
| $f'(x)$ | $+$ | $+$ | $-$ | $-$ | $+$ |
| $f(x)$ | $\nearrow$ | $\nearrow$ | $\searrow$ | $\searrow$ | $\nearrow$ |

6. Local Extrema (max./min.): Since $1-\sqrt{10}$ and $1+\sqrt{10}$ are in the domain, we conclude (from the result above), that $f(x)$ has a local max of $f(1-\sqrt{10})$ at $x = 1-\sqrt{10}$ and a local min of $f(1+\sqrt{10})$ at $x = 1+\sqrt{10}$.

7. Concavity:

$$f''(x) = \frac{(x^4 - 4x^3 - 12x^2 + 36x + 99)e^x}{(x^2-9)^3}$$

   Here you may use a graphing device (calculator or computer software) to realize that $f''(x)$ has no zero and the factor $x^4 - 4x^3 - 12x^2 + 36x + 99 > 0$. So the sign of $f''(x)$

entirely depends on the sign of $(x^2-9)^3$, which depends on the sign of $x^2-9$. So perform the sign test:

| $x$ | $(-\infty,-3)$ | $(-3,3)$ | $(3,+\infty)$ |
|---|---|---|---|
| $f''(x)$ | $+$ | $-$ | $+$ |
| $f(x)$ | $\cup$ | $\cap$ | $\cup$ |

8. Inflection Points: Since $-3$ and 3 are not in the domain of $f(x)$, the function has no inflection point (even though the concavity changes accross these values).

**Section 4.5 - Prob 64**

$$\lim_{x\to\infty}\frac{\ln x}{x^p} =^H \lim_{x\to\infty}\frac{\frac{1}{x}}{p\,x^{p-1}} = \lim_{x\to\infty}\frac{1}{p\,x^p} = 0\,.$$