Solution to "Paper" Homework #8

Section 4.6 - Prob 10

(b) Sketch of Figure.

(c) If we denote x as the side measurement of the cut out squares, then it is the same as the height of the box; let us also denote the width/length of the box as y. Then the volume is given by:

$$V = y^2 \cdot x\,.$$

(d) From the contruction of the box, we easily see that $y = 3 - 2x$.

(e)

$$V(x) = V = y^2 \cdot x = (3-2x)^2 \cdot x\,.$$

(f) Just like with any optimization problem, we first need to find the domain. To begin with, $x \geq 0$ (in real life, a negative measure of length does not make sense). We also require that the width/length of the box's base to be nonnegative, so $3 - 2x \geq 0$, which implies $x \leq \frac{3}{2}$. So together, the domain is $D = [0, \frac{3}{2}]$. Since the domain is a closed in interval, we use this method (which we learned in section 4.2, page 266):

- Compute $V'(x) = 3 \cdot (2x-3) \cdot (2x-1)$
- Find all the critical values: $V'(x) = 0$ implies $x = 3/2$ (which happens also to be the right end point of the domain) or $x = 1/2$.
- Evaluate:
  - $V(0) = 0$
  - $V(1/2) = 2$ (Maximum)
  - $V(3/2) = 0$

We conclude that the maximum volume we can have is 2 ft^3.

Section 4.6 - Prob 25 Suppose that the 10-meter-long wire is divide into two pieces. Let us denote x to be the length of the first piece, and $y = 10 - x$ of the second piece. Now, let us use the first piece for to make the square and the second piece for the equilateral triangle. So, the area of the square and triangle are given by

$$A_{\square} = \left(\frac{x}{4}\right)^2 \qquad \text{and} \qquad A_{\triangle} = \frac{\sqrt{3}}{4}\left(\frac{y}{3}\right)^2$$

respectively. So the total area is given by

$$A = \left(\frac{x}{4}\right)^2 + \frac{\sqrt{3}}{4}\left(\frac{y}{3}\right)^2 = \frac{x^2}{16} + \frac{\sqrt{3}}{36}(10-x)^2\,.$$

What is the domain? Again, remember that in reality, the sides of the square and triangle have nonnegative measurements, so we require that $x \geq 0$, and $y = 10 - x \geq 0 \Rightarrow x \leq 10$. So the domain of $A(x)$ is $D = [0, 10]$. Again, we have another closed interval, so we follow the steps of the previous problem:

- Compute the derivative:

$$A'(x) = \frac{4\sqrt{3}+9}{72}x - \frac{5\sqrt{3}}{9}.$$

- Find the critical values:

$$A'(x) = 0 \Rightarrow x = \frac{40\sqrt{3}}{4\sqrt{3}+9}.$$

- Compute:
  - $A(0) = \frac{25\sqrt{3}}{9} \approx 4.81$.
  - $A\left(\frac{40\sqrt{3}}{4\sqrt{3}+9}\right) \approx 3.23$ (Minimum).
  - $A(10) = \frac{25}{4} = 6.25$ (Maximum).

(a) To achieve a maximum total enclosed area, we use the whole wire for making the square.

(b) To achieve a minimum, we use exactly $\frac{40\sqrt{3}}{4\sqrt{3}+9}$ m for making the square.

Section 4.6 - Prob 31

$$P(R) = \frac{E^2 R}{(R+r)^2}$$

The Domain is $D = (0, +\infty)$.
We will use the First Derivative Test to find the extremum (you are welcome to use the Second Derivative Test, although the computation will be more complicated). Now, compute the derivative:

$$P'(R) = \frac{E^2 \cdot (r-R)}{(R+r)^3}.$$

There are two critical values: $R = -r$ (not accepted since it is outside the domain), and $R = r$. From here, you can use a sign test to easily see that $P(R)$ has an absolute maximum of $\frac{E^2}{4r}$ at $R = r$.

Section 4.7 - Prob 8

$$x^5 + 2 = 0, \quad x_1 = -1.$$

Let $f(x) = x^5 + 2$ (The problem is about approximating the root of $f(x)$). The Newton's Method requires the first derivative, so we compute: $f'(x) = 5x^4$. The form of iteration is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^5 + 2}{5x_n^4}.$$

So,

$$x_2 = x_1 - \frac{x_1^5 + 2}{5x_1^4} = -\frac{6}{5}.$$

$$x_3 = x_2 - \frac{x_2^5 + 2}{5x_2^4} \approx -1.1529.$$

Section 4.8 - Prob 44 Let us start with the first clue “constant acceleration a”:

$$a(t) = a\,.$$

Since the derivative of velocity is acceleration, to recover the velocity, we find the antiderivative of $a(t)$, which is the velocity, given by

$$v(t) = a \cdot t + C\,.$$

Use the clue “initial velocity v_o”:

$$v_o = v(0) = C\,.$$

Hence,

$$v(t) = a \cdot t + v_o\,.$$

Since the derivative of the displacement/distance/position is velocity, to recover it, we find the antiderivative of $v(t)$, which is:

$$s(t) = \frac{1}{2}a \cdot t^2 + v_o \cdot t + C\,.$$

Use the clue “initial displacement s_o”:

$$s_o = s(0) = C\,.$$

So,

$$s(t) = \frac{1}{2}a \cdot t^2 + v_o \cdot t + s_o\,.$$