DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS
MATH 121, SPRING 2014, MIDTERM EXAM

Your Name: Cuong Ngô **Instructor:** ____________

On this exam, you are not allowed to use a calculator, books or notes. For part B, it is not sufficient to just write down the answers. You must explain how you arrived at your answers and how you know they are correct.

Problem	Points	SCORE
1 - 6	60	
7	20	
8	20	
9	20	
10	20	
11	20	
12	20	
13	20	
TOTAL	200	

Part A - Multiple Choice Examination

Each right answer is worth 10 points

(60 points) *Select only* **one** *answer for each problem.*

$x^2-7x+12$
$=x^2-3x-4x+12$
$=x(x-3)-4(x-3)$
$=(x-4)(x-3)$

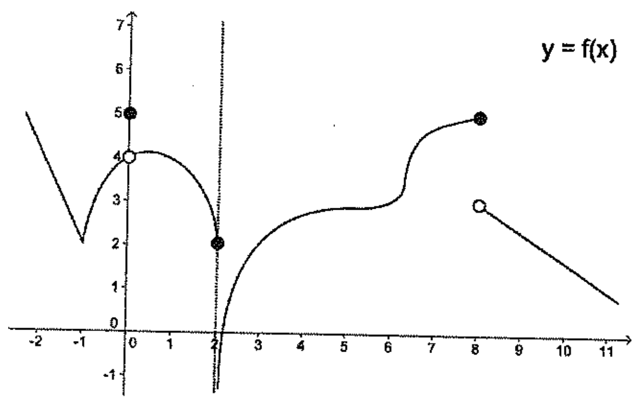
1. The domain of the function $f(x)=\frac{x+1}{\sqrt{x^2-7x+12}}$ is given by:

(A) $(0,1/2)$ (B) $(3,4)$ (C) $(-\infty,3]\cup[4,\infty)$
(D) $(-\infty,3)\cup(4,\infty)$ (E) $[-\infty,3)$ (F) None of the above is necessarily true

2. The inverse function for $f(x)=e^{x^3+1}$ is given by:
(A) $f(x)=e^{-(x^3+1)}$
(B) $f(x)=(\ln x-1)^{1/3}$
(C) $f(x)=(\ln x-1)^{-1/3}$
(D) $f(x)=e^{-x^3+1}$.
(E) $f(x)=(\ln x+1)^{1/3}$
(F) None of the above is true.

$y=e^{x^3+1}$
$\ln y=x^3+1 \Rightarrow x^3=\ln y-1$
$\Rightarrow x=(\ln y-1)^{1/3}$
$\Rightarrow y=(\ln x-1)^{1/3}$

3. Based on your observation only, decide whether each of the following statements is true or false for the function $y=f(x)$ graphed below.



T F f is differentiable at the point $x=-1$.
T F f is continuous at the point $x=-1$.
T F f is differentiable at the point $x=8$.
T F f is continuous at the point $x=2$.
T F $\lim_{x\to 0} f(x)=4$.

4. The function given by $f(x) = \begin{cases} cx^2 + 2x & \text{when } x < 2, \\ x^3 - cx & \text{when } x \geq 2 \end{cases}$

is continuous at $x = 2$:

(A) when $c = 2$
(B) when $c = 2/3$
(C) when $c = -2$
(D) never
(F) always

$\lim_{x \to 2^-} f(x) = c \cdot 4 + 4$
$\lim_{x \to 2} f(x) = 2^3 - c2 = 8 - 2c$

$4c + 4 = 8 - 2c$

$6c = 4 \Rightarrow c = \frac{4}{6} = \frac{2}{3}$

5. The limit

$$\lim_{h \to 0} \frac{5^{2+h} - 25}{h} = \lim_{h \to 0} \frac{5^{2+h} - 5^2}{h} = f'(2)$$
$= 5^2 \cdot \ln 5$
$= 25 \cdot \ln 5$

is equal to:

$f(x) = 5^x = e^{(\ln 5)x}$
$f'(x) = e^{(\ln 5)x} \cdot \ln 5$
$= 5^2 \cdot \ln 5$

(A) $5 \ln 5$
(B) $25 \ln 5$
(C) $25 \ln 25$
(D) 0
(F) None of the above.

6. Let f be a differentiable function, so that $f(1) = 2$, and $f'(1) = -5$. Let

$$g(x) = \frac{x^3 f(x)}{e^x + 1}$$

Find $g'(1)$.

$g'(x) = \frac{(x^3 f(x))'(e^x + 1) - (x^3 f(x))(e^x + 1)'}{(e^x + 1)^2}$

$3x^2 f(x) + x^3 f'(x)$

(A) $e - 1$
(B) $\dfrac{1-e}{(1+e)^2}$
(C) $\dfrac{1-e}{(1-e)^2}$
(D) $\dfrac{1+e}{(1+e)^2}$
(F) None of the above.

$= [(3x^2 f(x) + x^3 f'(x))(e^x + 1) - x^3 f(x) e^x] / (e^x + 1)^2$

$g'(1) = [(3 \cdot 2 + (-5))(e + 1) - 2 \cdot e] / (e + 1)^2$

$= (e + 1 - 2e) / (e + 1)^2$
$= (1 - e) / (e + 1)^2$

Part B - Essay questions

7. (20 points) Compute each of the following limits exactly or state DNE:

a. $\lim_{x\to -3} \frac{x^2-9}{x^2+2x-3} = \lim_{x\to -3} \frac{(x-3)\cancel{(x+3)}}{\cancel{(x+3)}(x-1)} = \frac{(-3)-3}{-3-1} = \frac{-6}{-4} = \frac{3}{2}$

b. $\lim_{h\to 0} \frac{(h-1)^2-1}{h} = \lim_{h\to 0} \frac{h^2-2h\cancel{+1}\cancel{-1}}{h} = \lim_{h\to 0}(h-2) = -2$

c. $\lim_{x\to -\infty} \frac{1-2x^2-x^4}{5+x-3x^4} = \lim_{x\to -\infty} \frac{\frac{1-2x^2-x^4}{x^4}}{\frac{5+x-3x^4}{x^4}}$

$= \lim_{x\to -\infty} \frac{\frac{1}{x^4} - 2\cdot\frac{1}{x^2} - 1}{\frac{5}{x^4} + \frac{1}{x^3} - 3} = \frac{-1}{-3} = \frac{1}{3}$

d. $\lim_{x\to\infty}(\sqrt{x^2+4x+1} - x\ )$

$= \lim_{x\to\infty} \frac{(\sqrt{x^2+4x+1}-x)(\sqrt{x^2+4x+1}+x)}{\sqrt{x^2+4x+1}+x}$

$= \lim_{x\to\infty} \frac{4x+1}{\sqrt{x^2+4x+1}+x} = \lim_{x\to\infty} \frac{4+\frac{1}{x}}{\sqrt{1+\frac{4}{x}+\frac{1}{x^2}}+1} = \frac{4}{\sqrt{1}+1} = 2$

8. **(20 points)** Find the derivative of the function by using logarithmic differentiation $f(x) = \frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5}$

$$\ln(f(x)) = \ln\left(\frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5}\right) = \ln[(x^2+1)^4] - \ln[(2x+1)^3(3x-1)^5]$$

$$\ln(f(x)) = 4\ln(x^2+1) - 3\ln(2x+1) - 5\ln(3x-1)$$

$$\Rightarrow [\ln(f(x))]' = 4[\ln(x^2+1)]' - 3[\ln(2x+1)]' - 5[\ln(3x-1)]'$$

$$\Rightarrow \frac{1}{f(x)} f'(x) = 4 \cdot \frac{1}{x^2+1}(2x) - 3 \cdot \frac{1}{2x+1} \cdot 2 - 5 \cdot \frac{1}{3x-1} \, 3$$

$$\Rightarrow \boxed{f'(x) = \frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5}\left(\frac{8x}{x^2+1} - \frac{6}{2x+1} - \frac{15}{3x-1}\right)}$$

9. **(20 points)** Use logarithmic differentiation to compute the derivative of the function $y = x^{\tan^{-1} x}$. Write the equation of the tangent line at $x = 1$.

$$\ln(y) = \ln(x^{\tan^{-1} x}) \Rightarrow \ln(y) = (\tan^{-1} x)\ln(x)$$

$$\Rightarrow \frac{d}{dx}\ln(y) = \frac{d}{dx}[(\tan^{-1} x)\ln(x)]$$

$$\Rightarrow \frac{dy}{dx} \cdot \frac{1}{y} = (\tan^{-1} x)' \ln(x) + (\tan^{-1} x) \cdot (\ln(x))'$$

$$\Rightarrow \frac{dy}{dx} = \left(x^{\tan^{-1} x}\right)\left(\frac{\ln(x)}{1+x^2} + \frac{\tan^{-1} x}{x}\right)$$

$$\left.\frac{dy}{dx}\right|_{x=1} = \underbrace{\left(1^{\tan^{-1}(1)}\right)}_{1}\left(\frac{\overset{\to 0}{\ln(1)}}{1+1^2} + \frac{\tan^{-1}(1)}{1}\right) = \frac{\pi}{4}$$

$$y - y_0 = m(x - x_0) \Rightarrow \boxed{y - 1 = \frac{\pi}{4}(x-1)}$$

10. **(20 points)** A particle moves along the path described by the parametric equations

$$x = e^t \qquad y = \sqrt{t} \qquad 0 \leq t \leq 1$$

a. Compute the derivative $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(\sqrt{t})}{\frac{d}{dt}(e^t)} = \frac{\frac{1}{2\sqrt{t}}}{e^t} = \frac{1}{2\sqrt{t} \cdot e^t}$$

b. Eliminate the parameter to get a Cartesian equation in x and y.

$$x = e^t \Rightarrow t = \ln x$$

$$y = \sqrt{t} \Rightarrow t = y^2$$

$$\Rightarrow y^2 = \ln x$$

11. (**20 points**) The position of a mass attached to a spring t seconds after being released is given by $f(t) = e^{-t/4}\cos(\pi t)$ feet.

(a) What is the **position** of the mass in the long run?

compute: $\lim_{t\to\infty} f(t) = \lim_{t\to\infty}\left(e^{-t/4}\cos(\pi t)\right)$

use squeeze theorem

$-1 \leqslant \cos(\pi t) \leqslant 1 \Rightarrow -e^{-t/4} \leqslant e^{-t/4}\cos(\pi t) \leqslant e^{-t/4}$

$\lim_{t\to\infty}\left(-e^{-t/4}\right) = 0 = \lim_{t\to\infty} e^{-t/4}$

$\Rightarrow \lim_{t\to\infty} e^{-t/4}\cos(\pi t) = 0$

(b) What is the **velocity** of the mass at $t = 4$ seconds? (You may leave your answer in terms of e.)

$v(t) = f'(t) = \left(e^{-t/4}\cos(\pi t)\right)'$

$= \left(e^{-t/4}\right)'\cos(\pi t) + e^{-t/4}\left(\cos(\pi t)\right)'$

$= \left(e^{-t/4}\right)\left(-\frac{1}{4}\right)\cos(\pi t) + e^{-t/4}\left(-\sin(\pi t)\right)\pi$

$\Rightarrow v(4) = f'(4) = \left(e^{-1}\right)\left(-\frac{1}{4}\right)\overbrace{\cos(4\pi)}^{1} + e^{-1}\left(\overbrace{-\sin(4\pi)}^{0}\right)\pi$

$= \frac{1}{e}\left(-\frac{1}{4}\right) = -\frac{1}{4e}$

12. **(20 points)**

The cost, in dollars, of producing x yards of a certain fabric is

$$C(x) = 1200 + 12x - 0.1x^2 + 0.0005x^3$$

Find the marginal cost function.

$C'(x) = 12 - 0.1 \cdot 2x + 0.0005 \cdot 3x^2$

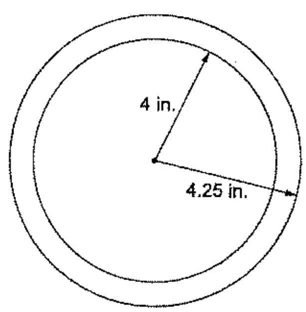
13. **(20 points)**

A cylindrical pipe has an outer radius of 4.25 inches and an inner radius of 4 inches. Estimate the cross-sectional (shaded) area of the pipe using differentials. For the final calculation, use $\pi = 3.14$ and the formula for the area of a circle $A = \pi r^2$.

$A(r) = \pi r^2$

$dA = A'(r)\,dr$

$= \pi \cdot 2r\,dr$



$r: 4 \longrightarrow 4.25$

$\Delta r = dr = 0.25$

$A(4.25) - A(4)$

$= \Delta A \approx dA = \pi \cdot 2 \cdot \underbrace{4 \cdot 0.25}_{1}$

$= 2\pi = 2 \cdot (3.14) = \boxed{6.28}$