MATH 121 - Fall 07

Review Problems for the Midterm Examination – Covers [1.1, 4.1] in Stewart

1. (a) Use the definition of the derivative to find $f'(3)$ when $f(x) = \dfrac{\pi}{1-2x}$.

(b) Find an equation of the tangent line at $(3, \dfrac{-\pi}{5})$.

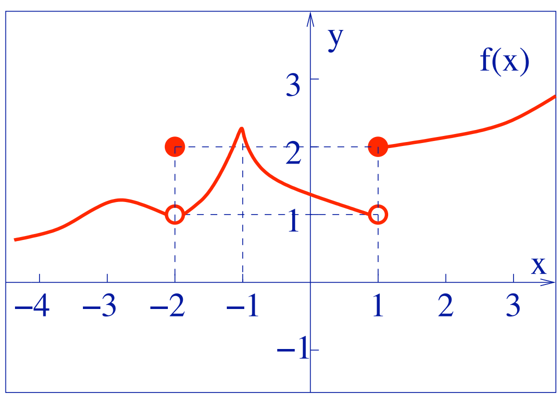
2. Consider the function $f(x) = 5x^5 + 4x^4 + 3x^3 + 2x^2 + x + 1$. Find an integer n such that $f(n) < 0$ and $f(n+1) > 0$. Show that there is a real number c such that $n < c < n+1$ and $f(c) = 0$ (name the theorem used).

3. The derivative of $y = f(x)$ at $x = 3$ is defined as

(A) $\displaystyle\lim_{h\to 0} \frac{f(3+x) - f(3)}{x}$ (B) $\displaystyle\lim_{x\to 0} \frac{f(3+h) - f(3)}{h}$ (C) $\displaystyle\lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$

(D) $\displaystyle\lim_{h\to 0} \frac{f(x+3) - f(3)}{h}$ (E) $\displaystyle\lim_{h\to 0} \frac{f(h+3) - f(3)}{x}$

4. Consider the function f defined by the following graph.



(a) f is not continuous at $x =$____________________________ .

(b) f does not have a derivative at $x =$____________________________.

(c) $\displaystyle\lim_{x\to -2} f(x) =$ ____________________________.

(d) $\displaystyle\lim_{x\to 1^-} f(x) =$ ____________________________.

(e) $f(-2) =$____________________________ .

5. Compute each of the following limits exactly (show your work) or state DNE:

(a) $\displaystyle\lim_{x\to 2} \frac{x^2 - 5x + 6}{x^2 - 4}$ (b) $\displaystyle\lim_{x\to 4} \sqrt{x+5} - x$ (c) $\displaystyle\lim_{x\to 0} \frac{\sqrt{x+2} - \sqrt{2}}{x}$

(d) $\lim_{x\to 1^-} \dfrac{\pi x}{(x-1)^2}$ (e) $\lim_{x\to\infty} \dfrac{1-2x+3x^2}{4x-7x^2}$ (f) $\lim_{t\to 0} \dfrac{(1+t)^{100}-1}{t}$

(g) $\lim_{x\to 2} \dfrac{x^2-x-2}{(x-2)^2}$ (h) $\lim_{x\to-\infty} \dfrac{2-x^2}{\pi x^2-3x+1}$ (i) $\lim_{x\to 0} \dfrac{1-\cos x}{\sin x}$

(j) $\lim_{x\to 3^-} [[x]]-x$ (Hint: Sketch the graph.) (k) $\lim_{h\to 0} \dfrac{\cos\ h-1}{h}$

(l) $\lim_{h\to 0} \dfrac{1-\cos\ h}{17h(\sin\ h)}$ (m) $\lim_{y\to 0} \dfrac{1-\cos\ y}{\pi y(\sin\ y)(1+\cos\ y)}$ (n) $\lim_{x\to 0} \dfrac{1-\cos(2x)}{17x^2}$

(o) $\lim_{x\to\infty} \dfrac{x^2(1+\sqrt{x})}{x(x-2x^{3/2})}$ (p) $\lim_{x\to 10^-} \sqrt{[[x^2]]}$ (q) $\lim_{x\to\pi} \dfrac{e^{\sin x}-1}{x-\pi}$

(r) $\lim_{\theta\to\frac{\pi}{3}} \dfrac{2\cos\theta-1}{\theta-\frac{\pi}{3}}$. (s) $\lim_{h\to 0} \dfrac{1}{h}\ln(\dfrac{17+h}{17})$ (t) $\lim_{x\to 0} \dfrac{\tan^2\ x+\sin\ x^2}{x^2+\sin^2\ x}$

(u) $\lim_{x\to 0} \dfrac{\frac{1}{x+1}-1}{x}$ (v) $\lim_{x\to\infty} \dfrac{\sqrt{x}-1}{x+\sqrt{x}}$

6. Find the interval(s) on which the function is continuous.

(a) $f(x)=\sqrt{x^2-4}$ (b) $g(x)=\dfrac{1}{x+1}$ (c) $y=\dfrac{\ln x}{\sqrt{1-x}}$ (d) $h(t)=\sqrt{16-t^2}$

7. Solve the equation for x.

(a) $e^{2x+1}=3$ (b) $\sqrt{e^{x^2+1}}=e^x$ (c) $\ln(x+1)=2$ (d) $\ln x+\ln(x-1)=\ln 2$

8. $\lim_{x\to 0^+} \dfrac{\cos\ x-\sin\ x}{x} =$ (A) $\dfrac{1}{2}$ (B) 1 (C) 2 (D) ∞ (E) $-\infty$ (F) DNE

9. $\lim_{x\to 1} (x-1)^{100}\left(\sin\dfrac{1}{x-1}\right)^{99} =$ (A) 0 (B) 1 (C) 2 (D) ∞ (E) $-\infty$ (F) DNE

10. Let $f(x)=\begin{cases} 2x-3 & x>1 \\ 1-2x & x<1 \end{cases}$. The function is not defined at $x=1$. Can a value of $f(1)$ be assigned to make f continuous at $x=1$? If so, give the value of $f(1)$. If not explain why not.

11. Let $f(x)=\begin{cases} x+1 & x\le 1 \\ 5+ax^2 & x>1 \end{cases}$. For which value of a will f be continuous at $x=1$?

12. Let $f(x)=\begin{cases} cx+3 & x\ge 1 \\ x-c & x<1 \end{cases}$. For which value of c will f be continuous at $x=1$?

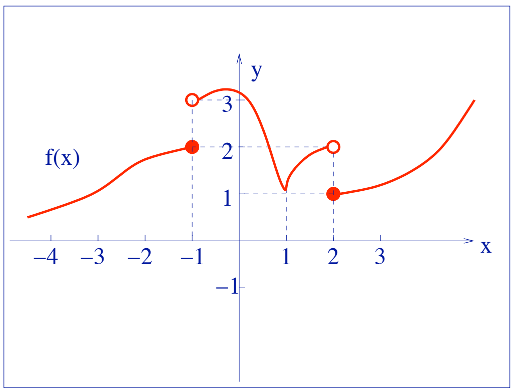
13. Let f be a function defined on $[0,100]$ such that $\lim_{x\to 17^-} f(x)=\lim_{x\to 17^+} f(x)=\pi$. Which of the following must be true?

(a) $f(17) = \pi$ (b) $f(\pi) = 17$ (c) $\lim_{x \to 17} f(x) = \pi$ (d) f is continuous at $x = 17$
(e) f is differentiable at $x = 17$.

14. The function $f(x)$ satisfies $f(4) = -489$ and $f'(4) = -378$. Which of the following values is the best linear approximation of $f(4.13)$?

(a) -2584.14 (b) 1072.14 (c) 973.86 (d) -538.14 (e) -1023

15. Consider the function f defined by the following graph. Put DNE (does not exist) if applicable.



(a) f is not continuous at $x =$_____ . (b) $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ does not exist at $x =$_____.

(c) $\lim_{x \to -1} f(x) =$ _____, (d) $\lim_{x \to 1^-} f(x) =$ _____. (e) $f(2) =$_____ .

16. Use the definition of the derivative to find the following: (write carefully and include all the steps.)

(a) $f'(x)$ when $f(x) = \dfrac{x}{x-1}$. (b) $g'(x)$ when $g(x) = \sqrt{x+1}$. (c) $k'(x)$ when $k(x) = \dfrac{17}{\sqrt{1-x}}$

17. At $x = 0$, the function given by $f(x) = \begin{cases} \sin\ x & x \leq 0 \\ x & x > 0 \end{cases}$ is

(a) undefined.

(b) continuous but not differentiable.

(c) differentiable but not continuous.

(d) neither continuous nor differentiable.

(e) both continuous and differentiable.

18. If $\lim_{x \to 2^+} f(x) = \pi$ and $\lim_{x \to 2^-} f(x) = 3.14$, then

(a) $\lim_{x \to 2} f(x) = \pi$ and f is continuous at $x = 2$.

(b) $\lim_{x\to 2} f(x) = \pi$ and f is not continuous at $x = 2$.

(c) $\lim_{x\to 2} f(x)$ does not exist and f is continuous at $x = 2$.

(d) $\lim_{x\to 2} f(x)$ does not exist and f is not continuous at $x = 2$.

(e) None of the above.

19. $\lim_{x\to 0} \dfrac{\tan\ x - \sin\ x}{x^3} =$ (a) $\dfrac{1}{2}$ (b) 1 (c) 2 (d) ∞ (e) DNE

20. If $f(3) = 0$, $f'(x) \geq 2$ for $2 < x < 4$, and $f''(x) \leq -2$ for $2 < x < 4$, then

(a) f is increasing and concave upward at $x = 3$.

(b) f is decreasing and concave upward at $x = 3$.

(c) f is increasing and concave downward at $x = 3$.

(d) f is decreasing and concave downward at $x = 3$.

(e) None of the above.

21. The following data is known about the function f and g.

t	$f(t)$	$f'(t)$	$g(t)$	$g'(t)$
0	0	6	0	-1
1	1	0	5	-2
3	3	2	1	-3
5	2	-1	3	7

Find the value of:

(a) $\dfrac{d}{dt}[f(t)g(t)]$ at $t = 3$.

(b) $\dfrac{d}{dt}\left(\dfrac{g(t)}{t^2}\right)$ at $t = 5$.

(c) $\dfrac{d}{dt}(e^t f(t))$ at $t = 0$.

(d) $\dfrac{d}{dt}((f(t) + g(t))$ at $t = 1$.

22. The derivative of $f(x) = \cos(x^2)$ at $x = 0$ is given by the limit

(a) $\lim_{h\to 0} \dfrac{\cos\ (h^2) - \cos\ (h)}{h}$ (b) $\lim_{h\to 0} \dfrac{\cos\ (h^2) - \sin\ (h^2)}{h}$

(c) $\lim_{h\to 0} \frac{\cos(h^2)-1}{h}$ (d) $\lim_{h\to 0} \frac{\cos(h)-1}{h}$ (e) None of the above

23 (a) Sketch the curve represented by the parametric equations $x = \ln t$, $y = t^2$ for $1 \le t \le 3$. Indicate with an arrow the direction in which the curve is being traced as t increases.

(b) Eliminate the parameter to find y as a function of x.

24. $\lim_{x\to\infty} \frac{(17x-\pi)(x-17)}{(\pi x-17)(17x-1)} =$ (a) 17 (b) π (c) $\frac{17}{\pi}$ (d) $\frac{\pi}{17}$ (e) $\frac{1}{\pi}$.

25. Let g be a function such that $g(x)$ is defined for all $x \neq \pi$. Also, suppose that g has a continuous second derivative for $x \neq \pi$. Also, $0 = g(0) = g(4)$, $g(-\sqrt{17}) = 3.2$, $g(-e) = 2.1$, $\lim_{x\to+\infty} g(x) = 2.3$, $\lim_{x\to-\infty} g(x) = -\infty$, $\lim_{x\to\pi^+} g(x) = -\infty$, $\lim_{x\to\pi^-} g(x) = +\infty$.

The following information is also known about g' and g''.

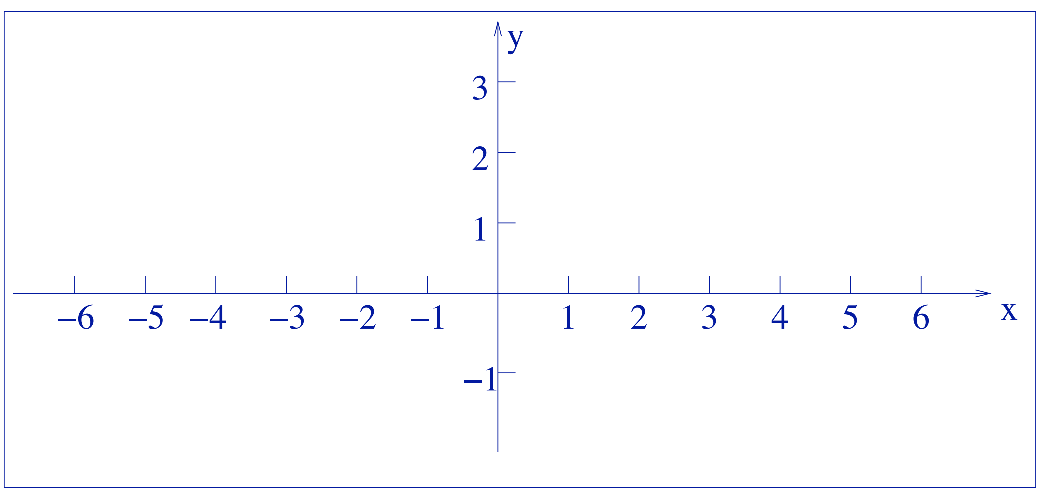
x	$x < -\sqrt{17}$	$-\sqrt{17} < x < -e$	$-e < x < 0$	$0 < x < \pi$	$\pi < x$
$g'(x)$	+	−	−	+	+
$g''(x)$	−	−	+	+	−

Find (no justification needed):

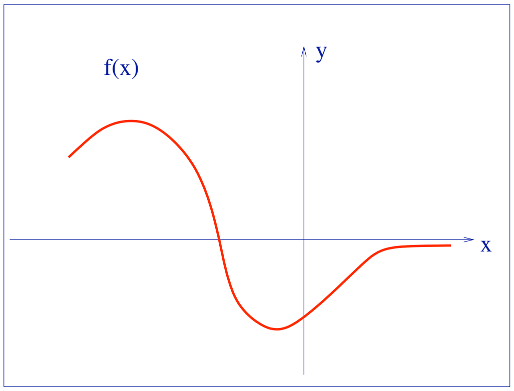
(a) The x coordinates of all local maxima. (b) The x coordinates of all local minima.

(c) The x coordinates of all inflection points. (d) The equations of all horizontal asymptotes.

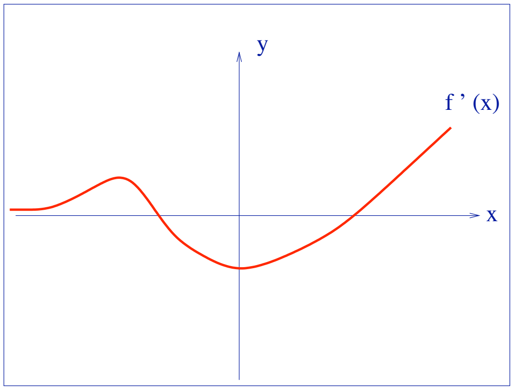
(e) The equations of all vertical asymptotes. (f) Carefully sketch a graph of g below:



26. The curve of function f is shown in the figure. On the same set of axes, sketch the graph of the derivative f'.



27. The curve of the derivative function $f'(x)$ is shown in the figure. Suppose $f(0) = 0$. On the same set of axes, sketch the graph of function $f(x)$.



28. (a) Sketch the curve given by the parametric equations $x = \sin t^2$, $y = 2\cos t^2$ for $0 \leq t \leq \sqrt{\pi}$. Indicate with an arrow the direction of the curve as t increases.

(b) Eliminate the parameter t to find an equation in x and y.

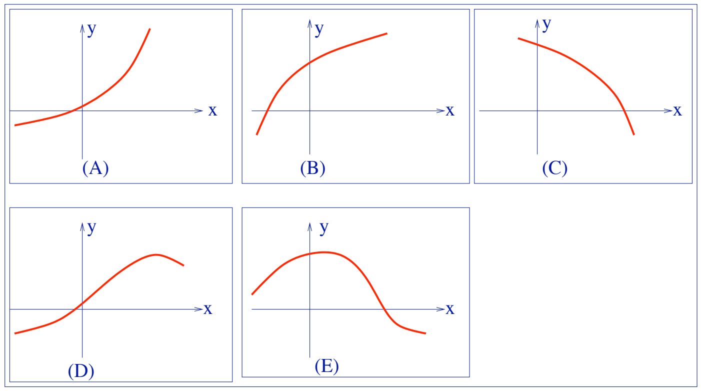
29. Consider the curve defined by $8x^2 - 5xy - y^3 = 80$. Find $\dfrac{dy}{dx}$ and an equation for the line tangent to the curve at the point $(4, 2)$

30. Let g and h be differentiable functions such that $g(1) = h(1) = 1, g'(1) = 3$, $h'(1) = -3$, $g'(2) = -4$, and $h'(2) = 5$. If $m(x) = g(h(x))$, then $m'(1) =$

(A)-9 (B)-4 (C) 0 (D) 12 (E) 15

31. If $f(x) = x^2$ and $g(x) = \ln x$, find the functions $f \circ g$, $f \circ f$, $g \circ f$, $g \circ g$ and their domains.

32. If f is a function of x such that $f'(x) > 0$ for all x and $f''(x) < 0$ for all x, which of the following could be a part of the graph of $y = f(x)$?



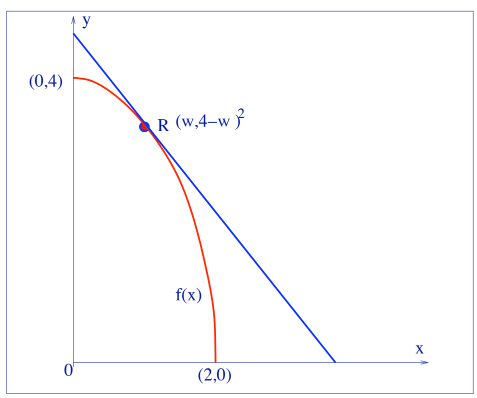
33. A particle moves along a line so that at any time t its position is given by $s(t) = 2\pi t + 2\cos(2\pi t)$.

(a) Find the velocity $v(t)$ at time t.

(b) Find the acceleration $a(t)$ at time t.

(c) Find the values of t, $0 \le t \le 1$, for which the particle is at rest.

34. Let $f(x) = 4 - x^2$. For $0 < w < 4$, let $A(w)$ be the area of the triangle formed by the coordinate axes and line tangent to the graph of f at the point $R = (w, 4 - w^2)$. (See the figure)



(a) Find the equation of the line tangent to the graph of f at the point R.

(b) Find an expression for $A(w)$.

(c) Find the rate of change of the area of the triangle with respect to w.

35. If $f(x) = \sqrt{x^3 + 1}$, find (a) $f(2)$, (b) $f^{-1}(2)$.

36. Consider the equation $xy = \sin\ y$. Find (a) $\dfrac{dy}{dx}$ and (b) an equation of the tangent line through the point $\left(\dfrac{4}{\pi\sqrt{2}}, \dfrac{\pi}{4}\right)$.

37. Given that $v(x) = u(w(x))$, find the six missing values in the below table:

x	$w(x)$	$w'(x)$	$u(x)$	$u'(x)$	$v(x)$	$v'(x)$
1	2	7	5	4	17	?
2	3	0	?	6	?	?
3	3	1	-4	?	?	8
5	11	2	2	-1	3	7

38. Consider the equation $xy = \tan y$. Find $\dfrac{dy}{dx}$ and an equation of the normal line to the graph at $\left(\dfrac{4}{\pi}, \dfrac{\pi}{4}\right)$.

39. Calculate the slope of the tangent line to the graph of $\cos(x^2 + y^2) = \sin x$ at the point $(0, \sqrt{\frac{\pi}{2}})$.

40. Consider the equation $\sin y = \cos x$. Find $\dfrac{d^2y}{dx^2}$ in terms of x and y.

41. A cyclist moving at 14 km/hr first rides north 8 km and then turns east. How fast is her distance from her starting point changing after one hour?

42. A ladder, 20 feet long, leans flush against a wall, i.e., the ladder is vertical. The foot of the ladder is pulled away from the wall at a constant velocity of $\frac{1}{2}$ ft/sec. How fast is the top of the ladder moving downward when the foot of the ladder is 9 feet from the wall?

43. Sketch a graph of a single function h that has all the following properties.

(a) $h(x)$ is defined and continuous for all $x \neq 0$

(b) $h'(x)$ exists for all $x \neq 0$

(c) $\lim\limits_{x \to -\infty} h(x) = 0^+$, $\lim\limits_{x \to \infty} h(x) = \infty$, $\lim\limits_{x \to 0^-} h(x) = -\infty$, $\lim\limits_{x \to 0^+} h(x) = 4$

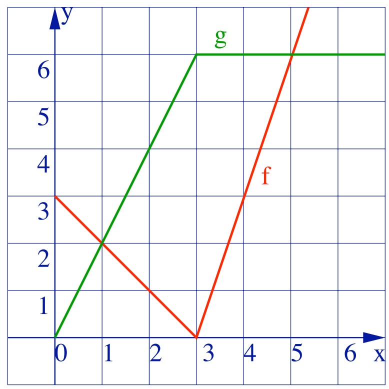
(d) $h(-2) = 2$, $h'(-2) = 0$, $h''(-2) < 0$, $h'(1) < 0$, $h''(x) > 0$ for all $x > 0$

44. Sketch the curve given by the parametric equations $x = t^2$, $y = t^3$ for $0 \leq t \leq 3$. Indicate with an arrow the direction of the curve as t increases. Eliminate the parameter t to find a Cartesian equation with x and y.

45. The potential energy V acting on a particle at a distance $q = 3$ is known to be $V(3) = -2.5$. If the derivative of V with respect to the variable q at $q = 3$ is $V'(3) = 0.6$, use linear approximation to determine the approximate value of the potential at $q = 2.7$.

46. If f and g are the functions whose graphs are shown, let $P(x) = f(x)g(x)$, $Q(x) = f(x)/g(x)$, and $C(x) = f(g(x))$. Find

(a) $P'(2)$ (b) $Q'(2)$ (c) $C'(2)$.



47. Find the point(s) on the curve

$$y = 1 - \frac{3}{x} + \frac{\pi}{x^2}$$

at which the tangent line is horizontal.

48. Find the point(s) on the curve

$$y = 2x - \tan\, x, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

at which the tangent line is parallel to $y = x$.

49. Let $g(x) = \dfrac{f(x)}{3x}$. Find $g'(-5)$ if $f'(-5) = 14$ and $f(-5) = -3$.

50. (a) Find an equation of the tangent line to the curve $y = e^x$ that is parallel to the line $x - 4y = 1$.
(b) Find an equation of the tangent line to the curve $y = e^x$ that passes through the origin.

51. Find $g'(x)$ if it is known that $\dfrac{d}{dx}(g(17x) + x^2) = x^5$.

52. A car is travelling at night along a highway shaped like a parabola with its vertex at the origin. The car starts at a point 100 m west and 100 m north of the origin and travels towards the origin. There is a statue located 100 m east and 50 m north of the origin (the statue is not on the highway). At what point on the highway will the car's headlights illuminate the statue?

53. If $y = \dfrac{\ln x}{x}$, then $\dfrac{dy}{dx} =$ (a) $\dfrac{1}{x}$ (b) $\dfrac{1}{x^2}$ (c) $\dfrac{\ln x - 1}{x^2}$ (d) $\dfrac{1 - \ln x}{x^2}$ (e) $\dfrac{1 + \ln x}{x^2}$

54. If $y = x^{(x^2)}$, for $x > 0$, then $\dfrac{dy}{dx} =$

(A) $x^2 \cdot x^{(x^2-1)}$ (B) $2x^{(x^2+1)} \cdot \ln x$ (C) $x + 2x \cdot \ln x$ (D) $x^{(x^2+1)}(1 + 2\ln x)$ (E) $3x^2$

55. $\dfrac{d}{dx}(x^{\ln x}) =$ (A) $x^{\ln x}$ (B) $(\ln x)^x$ (C) $\frac{2}{x}$ (D) $(\ln x)(x^{\ln x - 1})$ (E) $2(\ln x)(x^{\ln x - 1})$

56. Two cars start moving from the same point at the same time. One car travels north at 30 mph

and the other car travels east at 40 mph. How fast is the distance between the cars increasing two hours later?

57. A man starts walking north at 4 ft/sec from a point P. Five minutes later, a woman starts walking south at 5 ft/sec from a point 500 ft due east of P. At what rate are the walkers moving apart 15 minutes after the woman starts walking?

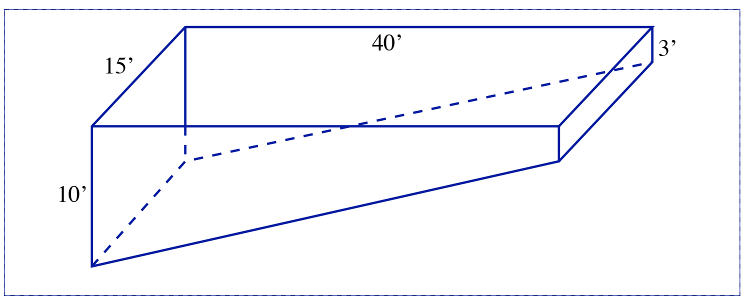
58. Find the point(s) on the curve

$$y = x^2 - 2x$$

at which the tangent line passes through the point $(1, -5)$

59. A runner runs around a circular track of radius of 100 meters at a constant speed of 7 m/sec. The coach is standing at a point 200 meters from the center of the track. The runner starts running at the point on the track nearest the coach. Determine how fast the distance between the coach and the runner is changing when the distance between them is 200 meters. (Hints: Arc length formula: s = rθ, Law of Cosines: $a^2 = b^2 + c^2 - 2bc\cos\theta$.)

60. A swimming pool is 15 feet wide, 40 feet long, 3 feet deep at one end, and 10 feet deep at the other end (see the figure). The pool is being filled with water at the rate of 10 ft^3/min. How fast the depth of the water is rising when the water is 4 feet deep?



61. For which of the following functions does the property $\dfrac{d^3y}{dx^3} = \dfrac{dy}{dx}$ hold?

I. $y = e^x$ II. $y = e^{-x}$ III. $y = \sin x$

(A) I only (B) II only (c) III only (D) I and II (E) II and III

62. Consider the curve described by $y = e^{xy}$. (a) Find $\dfrac{dy}{dx}$ in terms of x, y. (b) Find an equation of the tangent line to the curve at $x = 0, y = 1$.

63. The cost, in dollars, of producing x units of a certain commodity is

$$C(x) = 920 + 2x - 0.02x^2 + 0.00007x^3.$$

(a) Find the marginal cost function.

(b) Find $C'(100)$ and explain its meaning.

(c) Compare $C'(100)$ with the cost of producing the 101st unit.

64. The normal line to the curve represented by the equation $y = x^2 + 6x + 4$ at the point $(-2, -4)$ also intersects the curve at another point (x, y). Find the point (x, y). Show your work!

65. A particle moves along a line so that at any time t its position is given by

$$s(t) = 2\pi t + 2\cos(2\pi t), \qquad 0 \leq t \leq 1,$$

where t is measured in seconds and s in meters.

(a) Find the velocity $v(t)$ at time t.

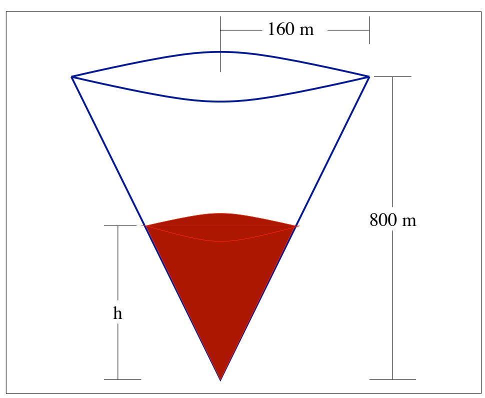
(b) Find the acceleration $a(t)$ at time t.

(c) When is the particle at rest?

(d) At $t = \dfrac{1}{6}$ sec, what is the position of the particle? Is it moving forward (in the positive direction)? Is it speeding up or slowing down?

66. Consider a differentiable function h with the properties that $h(17) = 22$ and $h'(17) = 2$. Then a linear approximation of $h(18.5)$ is (a) 17 (b) 22 (c) 28 (d) 23.5 (e)25.

67. The conical tank shown has a radius of 160 cm and a height of 800 cm. It is partially filled with water with the height of the water denoted by h.



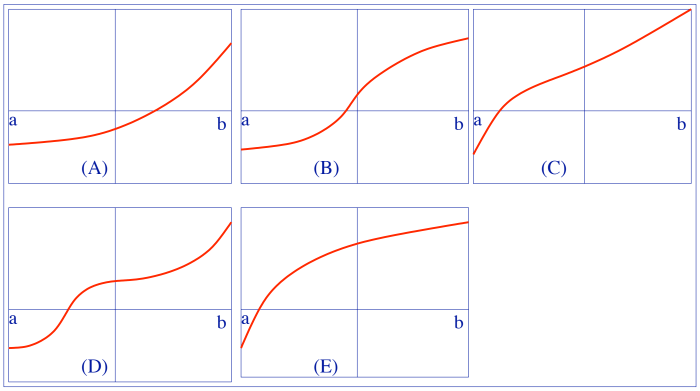
(a) Find the volume of the water as a function of h only.

(b) If water is leaking out of a hole at the vertex of the cone, what is the rate of change of the volume with respect to the height?

68. The large stucco teepee (north of Lawrence) needs painting. The height is 20 ft and the diameter is 40 ft. A coat of paint is $.02$ in thick. Use differentials to estimate the number of gallons required for one coat of paint. [Hint: 1 ft^3 = 7.48 gallons.]

69. A cubic block of ice is melting in such a way that each side is decreasing at the same rate. Use differentials to approximate the change in volume from when the side length changes from 5 cm to 4.9 cm.

70. Suppose f is a twice differentiable increasing function with exactly two points of inflection on $[a, b]$. Which of the following is a possible graph of f?

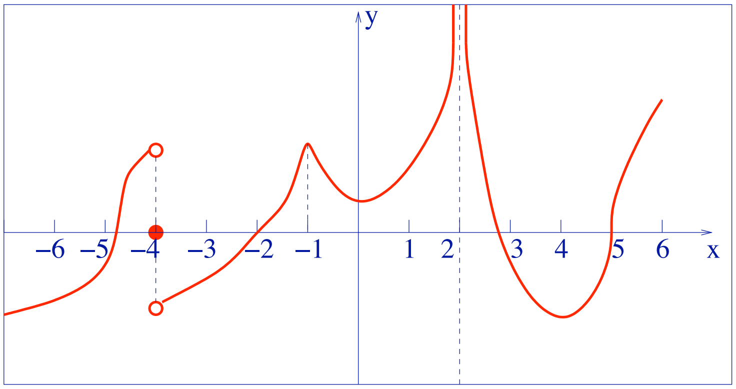


71. If a population of bacteria starts with 1000 bacteria and doubles every hour, then the number of bacteria after t hours is given by $N(t) =$1,000 2^t.

(a) When will the population reach 150,000?

(b) What will be the rate of growth of the bacteria population at that time?

72. Based on your observation only, decide whether each of the following statements is true or false for the function $y = f(x)$ graphed below.



T F (a) f is continuous at the point $x = -4$.

T F (b) f is differentiable at the point $x = -4$.

T F (c) f is continuous at the point $x = -1$.

T F (d) f is differentiable at the point $x = -1$.

T F (e) f is continuous at the point $x = 2$.

T F (f) f is differentiable at the point $x = 2$.

73. A heavy weight is thrown from a rooftop 160 feet high so that at t seconds, its height above the ground is $s(t) = -16t^2 + 48t + 160$ feet. Decide whether each of the following statements is true or false.

T F (a) The weight hits the ground after 2 seconds.

T F (b) When it hits the ground, its velocity is positive.

T F (c) Its acceleration is always positive.

T F (d) Its acceleration is always negative.

T F (e) Its acceleration is a constant.

74. A function f defined on the set of all real numbers has the following properties. If $x < 2$ or $3 < x$, then $f'(x) > 0$. If $2 < x < 3$, then $f'(x) < 0$. $f'(3) = 0$ but $f'(2)$ is undefined. If $x < 0$, then $f''(x) < 0$. If $0 < x < 2$ or $x > 2$, then $f''(x) > 0$. Decide whether each of the following statements is true or false.

T F (a) f is increasing on the interval $(2, 3)$.

T F (b) f is increasing on the interval $(-\infty, 2)$.

T F (c) f is concave down on the interval $(2, \infty)$.

T F (d) f is concave down on the interval $(-\infty, 0)$.

T F (e) f has a relative maximum at $x = 3$.

T F (f) f has a relative maximum at $x = 2$.

T F (g) f has a relative maximum at $x = -3$.

75. A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of 2 cm^3/s, how fast is the water level rising when the water is 5 cm deep?

76. A balloon is rising at a constant speed of 5 ft/s. A boy is cycling along a straight road at a speed of 15 ft/s. When he passes under the balloon it is 45 ft above him. How fast is the distance between the boy and the balloon increasing 3 s later?

77. A street light is mounted at the top of a 15 ft pole. A man walks away from the pole at the speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 40 ft from the pole?